

A Dirac type condition for properly coloured paths and cycles

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Abstract

Let c be an edge colouring of a graph G such that for every vertex v there are at least d different colours on edges incident to v . We prove that G contains a properly coloured path of length $2d$ or a properly coloured cycle of length at least $d + 1$. Moreover, if G does not contain any properly coloured cycle, then there exists a properly coloured path of length $3 \times 2^{d-1} - 2$.

1 Introduction

All graphs considered in this paper are simple without loops unless stated otherwise. Throughout this paper, G is assumed to be a graph. An *edge colouring* c of G an assignment of colours to the edges of G . An *edge-coloured graph* is a graph G with an edge colouring c of G .

An edge-coloured graph G is said to be *properly coloured*, or *p.c.* for short, if no two adjacent edges have the same colour. Moreover, G is *rainbow* if every edge has distinct colour. The *colour degree* $d^c(v)$ of a vertex v is the number of different colours on edges incident to v . The *minimum colour degree* $\delta^c(G)$ of a graph G is the minimal $d^c(v)$ over all vertices v in G . In this article, we study the p.c. paths and p.c. cycles in edge-coloured graphs G with $\delta^c(G) \geq 2$. For surveys regarding properly coloured subgraphs and rainbow subgraphs in edge-coloured graphs, we recommend Chapter 16 of [3] and [7] respectively.

Grossman and Häggkvist [6] gave a sufficient condition on the existence of p.c. cycles in edge-coloured graphs with two colours. Later on, Yeo [10] extended the result to edge-coloured graphs with any number of colours.

Theorem 1.1 (Grossman and Häggkvist [6], Yeo [10]). *Let G be a graph with an edge colouring c . If G has no p.c. cycle, then there is a vertex z in G such that no connected component of $G - z$ is joined to z with edges of more than one colour.*

Bollobás and Erdős [4] proved that if $n \geq 3$ and $\delta^c(K_n) \geq 7n/8$, then there exists a p.c. Hamiltonian cycle. (A path or cycle is Hamiltonian if it spans all the vertices.) They also asked the question of whether $\delta^c(K_n) \geq \lceil (n+5)/3 \rceil$ guarantees a p.c. Hamiltonian cycle. Fujita and Magnant [5] showed that $\delta^c(K_n) = \lfloor n/2 \rfloor$ is not sufficient by constructing an edge colouring c of K_{2m} with $\delta^c(K_{2m}) = m$, which has no p.c. Hamiltonian cycle. Alon

and Gutin [2] proved that for every $\epsilon > 0$ and $n > n_0(\epsilon)$ if no vertex in an edge-coloured K_n is incident with more than $(1 - 1/\sqrt{2} - \epsilon)n$ edges of the same colour, then there exists a p.c. Hamiltonian cycle. This easily implies that if $\delta^c(K_n) \geq (1/\sqrt{2} + \epsilon)n$ then there is a p.c. Hamiltonian cycle.

Li and Wang [9] proved that if $\delta^c(G) \geq d \geq 2$, then G contains a p.c. path of length $2d$ or a p.c. cycle of length at least $\lceil 2d/3 \rceil + 1$. We strengthen the bound of Li and Wang [9] to the best possible value. Our proof begins with the rotation-extension technique of Pósa [8], which we adapt for use on edge-coloured graphs.

Theorem 1.2. *Every edge-coloured graph G with $\delta^c(G) \geq 2$ contains a p.c. path of length $2\delta^c(G)$ or a p.c. cycle of length at least $\delta^c(G) + 1$.*

Note that a disjoint union of rainbow K_{d+1} has minimum colour degree d . The longest p.c. path and p.c. cycle have lengths d and $d + 1$ respectively. Together with the following example, we conclude that Theorem 1.2 is best possible.

Example 1.3. *For integers $p \geq d \geq 2$, define the edge-coloured graph $\tilde{G}(d; p)$ as follows: take a new vertex x and p vertex disjoint rainbow copies of K_d , H_1, H_2, \dots, H_p , add an edge of new colour c_j between x and every vertex of H_j for each j . It is easy to see that $\delta^c(\tilde{G}(d; p)) = d$, $\tilde{G}(d; k)$ has p.c. paths and p.c. cycles of lengths at most $2d$ and d respectively.*

In a non-edge-coloured graph G , it is a trivial fact that if $\delta(G) \geq 2$, then G contains a cycle. However, there exist edge-coloured graphs G with $\delta^c(G) \geq 2$ that do not contain any p.c. cycles, e.g. $\tilde{G}(2; p)$. Given an integer $k \geq 3$ and an edge colouring c of a graph G such that no p.c. cycle in G has length at least k , it is natural to ask what is the length of the longest p.c. path in G . We prove that the longest p.c. path grows exponentially with $\delta^c(G)$ for fixed k .

Theorem 1.4. *For integers $k \geq 3$, every edge-coloured graph G with $\delta^c(G) \geq \lceil 3k/2 \rceil - 3$ contains a p.c. path of length $k2^{\delta^c(G) - \lceil 3k/2 \rceil + 4} - 2$ or a p.c. cycle of length at least k .*

On the other hand, we show that there exist edge-coloured graphs G , which only contain p.c. paths and p.c. cycles of lengths at most $k2^{\delta^c(G) - k + 2} - 2$ and $k - 1$ respectively.

Proposition 1.5. *For integers $d \geq k - 1 \geq 2$, there exist infinitely many edge-coloured graphs G with $\delta^c(G) \geq d$ such that the longest p.c. paths and p.c. cycles in G are of lengths $k2^{d - k + 2} - 2$ and $k - 1$ respectively.*

For $k = 3$, Theorem 1.4 gives the following simple corollary, which is best possible by Proposition 1.5.

Corollary 1.6. *Every edge-coloured graph G with $\delta^c(G) = d$ contains a p.c. path of length $3 \times 2^{d-1} - 2$ or a p.c. cycle.*

For $d + 1 \geq k \geq 3$, we conjecture the following result.

Conjecture 1.7. *For integers $k \geq 3$, every edge-coloured graph G with $\delta^c(G) \geq k - 1$ contains a p.c. path of length $k2^{\delta^c(G) - k + 2} - 2$ or a p.c. cycle of length at least k .*

This conjecture is true for $k = 3$ and $k = d + 1$ by Corollary 1.6 and Theorem 1.2 respectively. Moreover, if Conjecture 1.7 is true for all $d + 1 \geq k \geq 3$, then it is best possible by Proposition 1.5.

We are also interested in the longest p.c. path in G with $\delta^c(G) = d$ without any constraint on p.c. cycles. Trivially, it has length at most d if G is a disjoint union of rainbow K_{d+1} . Thus, we may assume that G is connected. The following example shows that there are connected graphs with the longest p.c. paths of length $\lfloor 3d/2 \rfloor$.

Example 1.8. For integers $d \geq 1$ and $n \geq 3d/2$, define the edge-coloured graph $\widehat{G} = \widehat{G}(d; n)$ on n vertices as follows. Partition vertex set of G into X and Y with $X = \{x_1, x_2, \dots, x_d\}$. The subgraph induced by vertex set X is a rainbow K_d . The subgraph induced by vertex set Y is empty. For each $1 \leq i \leq d$, add an edge of new colour c_i between x_i and y for each $y \in Y$. By our construction, $\delta^c(\widehat{G}) = d$. Note that every p.c. path in \widehat{G} with both endpoints in Y must contain at least two vertices in X . Thus, every p.c. path in \widehat{G} is of length at most $|X| + \lfloor |X|/2 \rfloor = \lfloor 3d/2 \rfloor$.

We believe that the above example is best possible and conjecture the following.

Conjecture 1.9. Every edge-coloured graph G contains a p.c. Hamiltonian cycle or a p.c. path of length $\lfloor 3\delta^c(G)/2 \rfloor$.

This conjecture can be easily verified for $d \leq 3$. By case analysis, we can show that G contains a path of length $d + 2$ if $|G| \geq d + 2$ and $d \geq 4$. Therefore, the conjecture is true for $d \leq 5$. However, for $d \geq 6$ we are only able to show that G contains a p.c. path of length $6d/5 - 1$ or a p.c. Hamiltonian cycle.

Theorem 1.10. Every edge-coloured graph G contains a p.c. Hamiltonian cycle or a p.c. path of length $6\delta^c(G)/5 - 1$.

We set up the notations and tools in the next section. We prove Theorem 1.2 in Section 3. In Section 4, we prove Theorem 1.4 and Proposition 1.5. Theorem 1.10 is proved in Section 5. Finally, we consider a variant of colour degree in Section 6 and give a counterexample to a conjecture in [1].

2 Preliminaries

For $a, b \in \mathbb{N}$, let $[a, b]$ denote the set $\{i \in \mathbb{N} : a \leq i \leq b\}$. Write $[a]$ to be $[1, a]$.

For a graph G , $V(G)$ and $E(G)$ denote the sets of vertices and edges of G respectively. Denote the order of G by $|G|$. For a vertex subset $U \subset V(G)$, $G[U]$ is the induced (edge-coloured) subgraph of G on vertex set U . Given an edge-colouring c of G , a *colour neighbourhood* $N^c(v)$ of a vertex v is a maximal subset of the neighbourhood of v such that $c(v, w_1) \neq c(v, w_2)$ for distinct $w_1, w_2 \in N^c(v)$. Thus, $|N^c(v)| = d^c(v)$. It should be noted that there is a choice on $N^c(v)$, which we will specify later.

For convenience, let the vertices of G be labelled from 1 to $|G|$. A path P of length $l - 1$ is considered to be an l -tuple, (i_1, i_2, \dots, i_l) , where i_1, \dots, i_l are distinct. Similarly, a cycle of length l is considered to be an $(l + 1)$ -tuple, $(i_1, i_2, \dots, i_{l+1})$ with $i_1 = i_{l+1}$, where again i_1, \dots, i_l are distinct. For a p.c. path $P = (i_1, i_2, \dots, i_l)$ and $1 \leq j \leq l$, $N^c(i_j; P)$ is defined to be a colour neighbourhood of i_j chosen such that both i_{j-1} and i_{j+1} (if they exist) belong to $N^c(i_j; P)$. Again, there is still a choice on $N^c(i_j; P)$, which we will specify later. In other words, given a p.c. path $P = (i_1, i_2, \dots, i_l)$, the neighbours of i_j in P are always in $N^c(i_j; P)$ for $1 \leq j \leq l$.

We say that a p.c. path P has a *crossing* (with respect to some choices of $N^c(i_1; P)$ and $N^c(i_l; P)$) if there exist $1 \leq a < b \leq l$ such that $i_a \in N^c(i_l; P)$ and $i_b \in N^c(i_1; P)$. If $i_j \in N^c(i_l; P)$ and $c(i_{j-1}, i_j) \neq c(i_j, i_l)$, then $P' = (i_1, i_2, \dots, i_j, i_l, i_{l-1}, \dots, i_{j+1})$ is also a p.c. path. It is called a *rotation* of P with endpoint i_1 and pivot point i_j . A *reflection* of P is simply the p.c. path $(i_l, i_{l-1}, \dots, i_1)$. The set of p.c. paths that can be obtained by a sequence of rotations and reflections of P is denoted by $\mathcal{R}(P)$. We say P is *extensible* if there exists a vertex $j \notin V(P)$ such that (i_1, \dots, i_l, j) or (j, i_1, \dots, i_l) is a p.c. path. If P' is not extensible for every $P' \in \mathcal{R}(P)$, then P is said to be *maximal*. Note that if



Figure 1: Cycle $(1, 2, \dots, b, l, l-1, \dots, a, 1)$

P is maximal, $N^c(i_1; P) \cup N^c(i_l; P) \subset V(P)$ for every choice of $N^c(i_1; P)$ and $N^c(i_l; P)$. Hence, all maximal paths have length at least $\delta^c(G)$. We now study some basic properties of a p.c path P below.

Lemma 2.1. *Let c be an edge colouring of a graph G . Let $P = (1, 2, \dots, l)$ be a p.c. path. Suppose that there does not exist a p.c. cycle spanning $G[V(P)]$. Let $a \in N^c(1; P) \setminus \{2\}$ and $b \in N^c(l; P) \setminus \{l-1\}$ such that $b < a$, $c(1, a) \neq c(a, a+1)$ if $a < l$, and then $c(l, b) \neq c(b, b-1)$ if $b > 1$. Then, $C = (1, 2, \dots, b, l, l-1, \dots, a, 1)$ is a p.c. cycle, see Fig 1.*

Proof. Since $a \in N^c(1; P)$ and $b \in N^c(l; P)$, $c(1, 2) \neq c(1, a)$ and $c(l, l-1) \neq c(l, b)$. If $1 < b < a < l$, then C is a p.c. cycle. We may assume that $b = 1$. Moreover, $c(l, 1) = c(1, 2)$ or else $(1, 2, \dots, l, 1)$ is a p.c. cycle contradicting the assumption of $G[V(P)]$. Thus, $a < l$. Note that $c(l, l-1) \neq c(1, l) = c(1, 2) \neq c(1, a) \neq c(a, a+1)$ and so $C = (1, l, l-1, \dots, a, 1)$ is a p.c. cycle as required. \square

Lemma 2.2. *Let c be an edge colouring of a graph G with $\delta^c(G) \geq d \geq 2$. Let $P = (1, 2, \dots, l)$ be a p.c. path in G that is not extensible. Fix $N^c(1; P)$ and $N^c(l; P)$. Suppose P has a crossing with respect to $N^c(1; P)$ and $N^c(l; P)$. Furthermore, suppose that there does not exist a p.c. subgraph of $G[V(P)]$ consisting of a cycle C and a path Q such that*

- (i) $C = (i_1, i_2, \dots, i_p, i_1)$ with $p \geq d+1$,
- (ii) $Q = (i'_1, i'_2, \dots, i'_q)$,
- (iii) $V(C) \cap V(Q) = \emptyset$ and $V(P) = V(C) \cup V(Q)$,
- (iv) there exists $j \in [p]$ with $(i'_1, i_j) \in E(G)$ and $c(i'_1, i'_2) \neq c(i'_1, i_j)$.

Let $r = \min\{b \in N^c(l'; P)\}$. Then, there exists $s \in N^c(l; P)$ such that

- (a) $c(b, l) = c(b, b+1)$ for $b \in [r, s] \cap N^c(l; P)$,
- (b) for $b \in N^c(l; P)$ minimal such that $s < b$, $c(b, l) \neq c(b, b+1)$,
- (c) $c(1, a) = c(a, a+1) \neq c(a, a-1)$ for $a \in [r+1, s] \cap N^c(1, P) \setminus \{2\}$.

Moreover, if $s \geq 2$, then there exist $u, w \in N^c(1, P)$ such that $1 \leq r \leq s < u < w \leq l$ and the following statements hold:

- (d) $c(1, a) = c(a, a+1) \neq c(a, a-1)$ for $a \in [s+1, u] \cap N^c(1, P)$,
- (e) if $a \in N^c(1, P)$ and $a < w$, then $a \leq u$,
- (f) $c(1, w) \neq c(w, w+1)$.

Proof. Write $N^c(1) = N^c(1; P)$ and $N^c(l) = N^c(l; P)$. Since P is a non extensible p.c. path, $N^c(1) \cup N^c(l) \subset V(P)$ and $l \geq d+1$. Note that $c(l, r) = c(r, r+1)$, or else $C = (r, r+1, \dots, l, r)$ is a p.c. cycle containing $N^c(l) \cup \{l\}$ of length at least $d+1$. In addition, $Q = (r-1, r-2, \dots, 1)$ is a p.c. path, which contradicts the assumption of $G[V(P)]$. Let $s \in N^c(l)$ be maximal such that $c(l, b) = c(b, b+1)$ for $b \leq s$ and $b \in N^c(l)$. Observe that s exists and $r \leq s$. Hence, (a) and (b) hold.

If (c) is false, then there exists $a \in N^c(1) \cap [r+1, s] \setminus \{2\}$ such that $c(1, a) \neq c(a, a+1)$. Let $b \in N^c(l)$ be maximal such that $b < a$. Note that $a > 2$ and $b < s < l-1$. By (a) and Lemma 2.1, $C = (1, 2, \dots, b, l, l-1, \dots, a, 1)$ is a p.c. cycle containing $N^c(1) \cup \{l\}$ and $Q = (b+1, b+2, \dots, a-1)$ is a p.c. path. This is a contradiction, so (c) is true.

Now assume that $s \geq 2$. Let $a' \in N^c(1)$ be maximal. Recall that $d \geq 2$, so $a' > 2$. Note that $c(1, a') = c(a', a'-1)$ or else the cycle $C = (1, 2, \dots, a', 1)$ and path $Q = (a'+1, \dots, l)$ are both p.c. contradicting the assumption of $G[V(P)]$. By (c), $s < a'$. Let $a'' \in N^c(1)$ be minimal such that $a'' > s \geq 2$. If $c(1, a'') = c(a'', a''+1)$, then the cycle $C = (1, 2, \dots, s, l, l-1, \dots, a'', 1)$ and $Q = (s+1, s+2, \dots, a'')$ are p.c. by (a) and Lemma 2.1. Moreover, $C \supset N^c(1) \cup \{1\}$, so $|C| \geq d+1$, a contradiction. Thus, we may assume that $c(1, a'') = c(a'', a''+1)$. Let $u \in N^c(1)$ be maximal such that $c(1, a) = c(a, a+1)$ for $a \in N^c(1)$ and $s < a \leq u$. Thus, (d) is true by construction. Recall that $c(1, a') = c(a'-1, a')$, so $u < a'$. Let $w \in A$ be minimal such that $u < w$. Thus, (e) and (f) easily follow. Therefore, the proof of the lemma is completed. \square

3 Maximal p.c. paths with crossings

In this section, we show that for every maximal p.c. path P that has a crossing, there exists a cycle of length $\delta^c(G) + 1$ unless $|P| \geq 2\delta^c(G) + 1$. We split the cases when $\delta^c(G) = 2$ and $\delta^c(G) \geq 3$ separately.

Lemma 3.1. *Let c be an edge colouring of a graph G such that $\delta^c(G) \geq 2$. Let P be a maximal p.c. path with end points x and y . Suppose P has a crossing with respect to some $N^c(x; P)$ and $N^c(y; P)$. Then, there exists a p.c. cycle C in $G[V(P)]$.*

Lemma 3.2. *Let c be an edge colouring of a graph G such that $\delta^c(G) = d \geq 3$. Let $P = (1, 2, \dots, l)$ be a maximal p.c. path. Suppose P has a crossing with respect to some $N^c(1; P)$ and $N^c(l; P)$. Then, there exists a p.c. subgraph of $G[V(P)]$ consisting of a cycle C and a path Q such that*

- (i) $C = (i_1, i_2, \dots, i_p, i_1)$ with $p \geq d$,
- (ii) $Q = (i'_1, i'_2, \dots, i'_q)$,
- (iii) $V(C) \cap V(Q) = \emptyset$ and $V(P) = V(C) \cup V(Q)$,
- (iv) there exists $j \in [p]$ with $(i'_1, i_j) \in E(G)$ and $c(i'_1, i'_2) \neq c(i'_1, i_j)$.

Moreover, $p \geq d+1$ unless $|P| \geq 2d+1$.

In Lemma 3.1, i.e. when $\delta^c(G) = 2$, we only show the existence of a p.c. cycle C in $G[V(P)]$. In Lemma 3.2, i.e. when $\delta^c(G) \geq 3$, we further show that there is a spanning p.c. path Q in $G[V(P) \setminus V(C)]$, if $V(P) \setminus V(C) \neq \emptyset$. Next, we show that Lemma 3.1 and Lemma 3.2 imply Theorem 1.2.

Proof of Theorem 1.2. Let P be a maximal p.c. path in G . Without loss of generality, $P = (1, 2, \dots, l)$. Fixed $N^c(1; P)$ and $N^c(l; P)$. If P has a crossing, then we are done by Lemma 3.1 and Lemma 3.2. If P does not have a crossing, then $|(N^c(1; P) \cup \{1\}) \cap (N^c(l; P) \cup \{l\})| \leq 1$ and so $|P| \geq |(N^c(1; P) \cup \{1\}) \cup (N^c(l; P) \cup \{l\})| \geq 2d+1$. \square

First we prove Lemma 3.1, that is the case when $\delta^c(G) = 2$.

Proof of Lemma 3.1. Suppose the lemma is false. Let G be a graph with an edge colouring c containing a maximal p.c. path $P = (1, 2, \dots, l)$ that contradicts Lemma 3.2. Fixed

$N^c(i; P)$ for $i \in [l]$ such that P has a crossing. Note that $|N^c(i; P) \cap V(P)| \geq 2$ for $i \in [l]$. The induced subgraph $H = G[V(P)]$ does not contain any p.c. cycle and $\delta^c(H) \geq 2$. By Theorem 1.1, there exists a vertex z in H such that no connected component of $H - z$ is joined to z with edges of more than one colour. However, H is 2-connected as P has a crossing. This contradicts the existence of such z as $\delta^c(H) \geq 2$. \square

Next, we prove Lemma 3.2. We now sketch the proof of the first assertion of Lemma 3.2. Let G be a graph with an edge colouring c . Let $P = (1, 2, \dots, l)$ be a maximal p.c. path in G . Let C be a p.c. cycle such that

$$V(C) \subset [l] \text{ and } [l] \setminus V(C) \text{ is an interval,} \quad (1)$$

$[l] \setminus V(C) = [i'_1, i'_l]$ say. By taking $Q = (i'_1, i'_1 + 1, \dots, i'_l)$ or $Q = (i'_l, i'_l - 1, \dots, i'_1)$, C and Q satisfy properties (ii) – (iv) in Lemma 3.2. Thus, it suffices to show that there exists a p.c. cycle C satisfying (1) with $|C| \geq d$. Suppose the lemma is false. Let P be a maximal p.c. path in G contradicting the lemma. We then show that there exist $1 \leq r \leq s < u < w \leq l$ satisfying the conditions of Lemma 2.2. Then, $C = (1, 2, \dots, s, l, l-1, \dots, w, 1)$ is a p.c. cycle by Lemma 2.1 satisfying (1), see Figure 2. By assuming that $|[r, s] \cap N^c(l; P)|$ is maximal, we then deduce that $|C| \geq d$, Claim 3.3. Thus, the first assertion of Lemma 3.2 holds. A detailed analysis of $N^c(i; P)$ for $i \in [l]$ is needed in order to prove the second assertion.

Proof of Lemma 3.2. Suppose the lemma is false. Let G be an edge-minimal graph with an edge colouring c containing a maximal p.c. path $P = (1, 2, \dots, l)$ that contradicts Lemma 3.2. By the discussion above, in order to prove the first assertion of the Lemma 3.2, it suffices to show that there exists a p.c. cycle C satisfying (1) with $|C| \geq d$. Similarly, if $|C| \geq d+1$, then the second assertion of the lemma also holds. Pick $N^c(1; P)$ and $N^c(l; P)$ such that P has a crossing. Let

$$\begin{aligned} A &= \{a_i : 1 \leq i \leq d_1\} = N^c(1; P) \text{ and} \\ B &= \{b_j : 1 \leq j \leq d_l\} = N^c(l; P), \end{aligned}$$

where both $(a_i)_{i=1}^{d_1}$ and $(b_j)_{j=1}^{d_l}$ are increasing sequences. By maximality of P and choices of $N^c(i; P)$, we have

$$d^c(1) = d_1, \quad d^c(l) = d_l, \quad a_1 = 2 \text{ and } b_{d_l} = l - 1.$$

If $l \in A$ and $1 \in B$, then $(1, 2, \dots, l, 1)$ is a cycle of length $l \geq d+1$. Hence,

$$l \notin A \text{ or } 1 \notin B, \quad (2)$$

so $l \geq d+2$. Let $r = r(P) = b_1$. There exists $s = s(P) \in B$ satisfying the conditions (a), (b) and (c) of Lemma 2.2. Let $S = S(P) = [r, s] \cap B$. Also, note that

$$c(1, a_{d_1}) = c(a_{d_1}, a_{d_1} - 1) \quad (3)$$

or else $(1, 2, \dots, a_{d_1}, 1)$ is a p.c. cycle satisfying (1) with $|C| \geq d+1$ as $A \cup \{1\} \subset V(C)$. We further assume that $|S| = |S(P)| \geq |S(P')|$ for $P' \in \mathcal{R}(P)$, i.e. $|S|$ is maximal. If $|S| \geq 2$, then $s \geq 2$. If $|S| = 1$ and $s = 1$, then $l \notin A$ by (2). Note that $|S(P')| \geq 1$ for $P' \in \mathcal{R}(P)$ as $r(P') \in S(P')$. By replacing P with $(l, l-1, \dots, 1)$, we may assume that

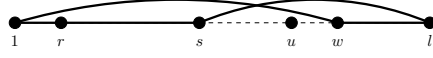


Figure 2: $C = (1, 2, \dots, s, l, l-1, \dots, w, 1)$



Figure 3: $P' = (r+1, \dots, l, r, \dots, 1)$

$s \geq 2$ as $l \notin A$ by (2). Thus, we can find $u, w \in A$ satisfying the conditions (d), (e) and (f) of Lemma 2.2. Note that $w \geq 3$ and $s < l-1$. By Lemma 2.1 and Lemma 2.2 (a) and (f),

$$C = (1, 2, \dots, s, l, l-1, \dots, w, 1),$$

again see Figure 2, is a p.c. cycle satisfying (1). Moreover,

$$|C| = |[1, s] \cup [w, l]| = |A| + 1 + |[2, s] \setminus A| + |[w, l] \setminus A| - |[s+1, u] \cap A|. \quad (4)$$

In the next claim, we prove that $|C| \geq d$. Hence, the first assertion of Lemma 3.2 holds.

Claim 3.3. (a) We have $|C| \geq d$. Moreover, the first assertion of Lemma 3.2 holds by taking $Q = (s+1, s+2, \dots, w-1)$.

(b) If the second assertion of Lemma 3.2 is false for P , then

$$\begin{aligned} |C| &= d, \quad |A| = d, \quad S = [r, s], \quad \text{and} \\ A &= \begin{cases} [2, r] \cup [t, u] \cup [w, l] & \text{if } r \neq 1, \\ \{2\} \cup [t, u] \cup [w, l-1] & \text{if } r = 1, \end{cases} \end{aligned} \quad (5)$$

where $t = u - |S| + 1 \geq 3$. Also, for $t \leq i \leq u$, $c(1, i) = c(i, i+1)$.

Moreover, if $P' = (i_1, \dots, i_l) \in \mathcal{R}(P)$ has a crossing and $|S(P')| = |S|$ for some $N^c(i_1; P')$ and $N^c(i_l; P')$ and $s(P') \neq i_1$, then the corresponding statements hold (by the map $i_j \rightarrow j$).

Proof of claim. Recall that $r \in B$, so $c(l, l-1) \neq c(l, r) = c(r, r+1)$ by Lemma 2.2 (a). In addition, if $r \neq 1$, then $c(r, l) \neq c(r, r-1)$. Thus, the path $P' = (r+1, r+2, \dots, l, r, r-1, \dots, 1)$, see Figure 3, is a member of $\mathcal{R}(P)$. Since $c(r+1, r) \neq c(r+1, r+2)$, we choose $N^c(r+1; P')$ such that $r \in N^c(r+1; P')$. Set $N^c(1; P') = A$ if $r \neq 1$ and $N^c(1; P') = A \cup \{l\} \setminus \{2\}$ if $r = 1$. Observe that $u \in N^c(1; P')$. Thus, P' has a crossing. If $a \in A \cap [r+1, u] \setminus \{2\}$, then $c(1, a) = c(a, a+1)$ by Lemma 2.2 (c) and (d). Hence, $S(P') \supset A \cap [r+1, u] \setminus \{2\}$. Since $|S|$ is maximal,

$$|S| \geq |S(P')| \geq |A \cap [r+1, u]| - \delta_{1,r}, \quad (6)$$

where $\delta_{1,r} = 1$ if $r = 1$, and $\delta_{1,r} = 0$ otherwise. Recall by (2) that $1 \notin B$ or $l \notin A$, so $|\{l\} \setminus A| - \delta_{1,r} \geq 0$. By adding (4) and (6), we have

$$\begin{aligned} |S| + |C| &\geq |A| + 1 + |[2, r] \setminus A| + |[w, l] \setminus A| - \delta_{1,r} + |[r+1, s]| \\ &\geq |A| + 1 + |\{l\} \setminus A| - \delta_{1,r} + |S \setminus \{r\}| \\ &\geq |A| + |S| \geq d + |S|. \end{aligned} \quad (7)$$

This implies $|C| \geq d$, so (a) holds.

Suppose that the second assertion of Lemma 3.2 is false. Hence, $|C| \leq d$. In fact $|C| = d$, so all inequalities in (7) are actually equalities. Thus, $S = [r, s]$, $|A| = d$ and $[2, r] \cup [w, l - \delta_{1,r}] \subset A$. Equality also holds in (6), so $|S| = |S(P')|$. Thus, $S(P') = A \cap [r + 1, u] \setminus \{2\}$. By replacing P with P' , we deduce that $S(P')$ is also an interval. Hence, $A \cap [r + 1, u] = [t, u]$ with $t = u - |S| + 1 \geq 3$. Therefore, A satisfies (5). The last assertion follows easily by replacing P with P' . This completes the proof of the claim. \square

Recall that P is a counterexample, so $|P| \leq 2d$. Since P is a maximal p.c. path, every $P' \in \mathcal{R}(P)$ has a crossing. Next, we show that $|S| \geq 2$.

Claim 3.4. $|S| \geq 2$.

Proof of claim. Suppose $|S| = 1$. Recall that we assume that $s \geq 2$, so $r = s \geq 2$. Thus,

$$A = [2, r] \cup \{u\} \cup [w, l]$$

by (5). Recall that $l \geq d + 2$ by the remark after (2). If $l - 1 \in A$, then $c(1, l - 1) \neq c(l - 1, l - 2)$ (as $|S(l, l - 1, \dots, 1)| = 1$) and so $(1, 2, \dots, l - 1, 1)$ is a p.c. cycle of length $l - 1 \geq d + 1$, a contradiction. Thus, $l - 1 \notin A$ and $w = l$. Since $|A| = d$, $a_{d-1} = u$. By Lemma 2.2 (d), $c(1, u) = c(u, u + 1) \neq c(u, u - 1)$ and so $(1, 2, \dots, u, 1)$ is a p.c. cycle. Thus, $u = d$ and $A = [2, d] \cup \{l\}$. We deduce that $r = d - 1$ and $b_2 \geq d$. Recall that $c(1, l) = c(l, l - 1) \neq c(l, b_2)$. Hence, $c(b_2, l) = c(b_2, b_2 - 1)$ or else $(1, 2, \dots, b_2, l, 1)$ is a p.c. cycle of length at least $d + 1$ satisfying (1), a contradiction. If $b_2 = d$, then $c(l, d - 1) = c(d - 1, d) = c(l, d)$, a contradiction. If $b_2 = d + 1$, then $(1, d, d + 1, l)$ is a monochromatic path of length 3. Let G' be the edge-coloured subgraph of G obtained by remove the edge $(d, d + 1)$. Note that $\delta^c(G') = d$. The p.c. path

$$P' = (d, 1, 2, \dots, d - 1, l, l - 1, \dots, d + 1)$$

can be obtained by a rotation of P with pivot point $d - 1$ and endpoint 1 followed by a rotation with pivot point d and endpoint d . Hence, P' is maximal in G and also in G' and so G' contradicts the edge-minimality of G . Therefore, $B \cap \{d, d + 1\} = \emptyset$ and so $B \cup \{d - 1\} \cup [d + 2, l - 1]$. Hence, $l \geq 2d + 1$, a contradiction. This completes the proof of Claim 3.4. \square

Since $|S| \geq 2$, (5) implies that $d \geq 4$. Also, $s(P') \geq 2$ for all $P' \in \mathcal{R}(P)$ that has a crossing. In the next claim, we show that if necessary t may be assumed to be at least $r + 3$.

Claim 3.5. *We may assume that $t \geq r + 3$ if necessary.*

Proof of claim. Suppose the contrary, so either $t = r + 1$ or $t = r + 2$. Recall that $|S| \geq 2$ and $S = [r, s]$ by Claim 3.3 (b), $t - 1 \in S$. Without loss of generality, $r = 1$ otherwise consider the path $P' = (t, t + 1, \dots, l, t - 1, t - 2, \dots, 1)$ instead as $|S(P')| = |[t, u]| = |S|$ by Claim 3.3 (b). Hence, $t = 3$, $S = [1, s]$ and $s = |S|$. Moreover, by Claim 3.3 (b) we know that

$$A = [2, s + 2] \cup [w, l - 1] \text{ and } c(1, i) = c(i, i + 1) \text{ for } 3 \leq i \leq s + 2. \quad (8)$$

Define ϕ to be the permutation on $[l]$ such that

$$(\phi(1), \phi(2), \dots, \phi(l)) = (3, 4, \dots, l, 2, 1).$$

We are going to show that the following statements are true for $0 \leq i \leq (l - 1)/2$ (subjecting to some choices of the colour neighbours which will come clear):

- (i) $P_i = (\phi^i(1), \phi^i(2), \dots, \phi^i(l)) \in \mathcal{R}(P)$,
- (ii) $S(P_i) = \{\phi^i(j) : 1 \leq j \leq s\}$,
- (iii) $N^c(\phi^i(1); P_i) = A_i$, where $A_i = \{\phi^i(j) : j \in A\}$,
- (iv) for $3 \leq j \leq s+2$,

$$c(\phi^i(1), \phi^i(j)) = c(\phi^i(j), \phi^i(j+1)) \neq c(\phi^i(j), \phi^i(j-1)),$$

- (v) $N^c(\phi^i(2); P_i) = \{\phi^i(j) : j \in \{1\} \cup [3, d+1]\}$, and
- (vi) for $4 \leq j \leq d+1$,

$$c(\phi^i(2), \phi^i(j)) = c(\phi^i(j), \phi^i(j+1)) \neq c(\phi^i(j), \phi^i(j-1)).$$

First, we are going to show that (i) – (iv) are true by induction on i . It is true for $i = 0$ by (8), so we may assume that $i \geq 1$ and (i) – (iv) hold for $i - 1$. For simplicity, we may assume that $i = 1$ by considering the map $\phi^{i-1}(j) \mapsto j$. Since $2 \in S(P_0)$, $c(1, 2) \neq c(2, 3) = c(2, l)$. Note that $P_i = (\phi(1), \phi(2), \dots, \phi(l))$ can be obtained by a rotation P_0 with pivot point 2 and endpoint 1 and a reflection. Thus, $P_i \in \mathcal{R}(P)$ and so (i) is true. Set $N^c(\phi(l); P_1) = N^c(1; P_0) = A_0$ by (iii). Observe that

$$\{\phi(j) : j \in [s]\} = [3, s+2] \subset N^c(\phi(l); P_1),$$

so $\{\phi(i) : i \in [s]\} \subset S(P_1)$ by (iv). Recall that $|S|$ is maximal, so $|S(P_1)| = |S|$ and (ii) holds. Note that P_i has a crossing as $\phi(1) = 3 \in A_0 = N^c(\phi(l); P_1)$ by our choice of $N^c(\phi(l); P_1)$. By Claim 3.3 (b),

$$N^c(\phi(1); P_1) = \{\phi(j) : j \in \{2\} \cup [t_1, t_1 + s - 1] \cup [w, l - 1]\}$$

for some $t_1 \geq 3$. In addition,

$$c(\phi(1), \phi(j)) = c(\phi(j), \phi(j+1)) \neq c(\phi(j), \phi(j-1))$$

for $t_1 \leq j \leq t_1 + s - 1$. If $t_1 > 3$, then Claim 3.5 is true by taking $P = P_i$, a contradiction. Thus, $t_1 = 3$ and $N^c(\phi(1); P_1) = A_1$, so both (iii) and (iv) are true. Therefore, (i) – (iv) are true for all $i \geq 0$.

Next, we show that (i) – (iv) implies (v) and (vi). For simplicity, we may assume that $i = 0$ by considering the map $\phi^i(j) \mapsto j$. Since $1 = r \in S(P_0)$ by (ii), the path $P' = (2, 3, \dots, l, 1)$ is also a member of $\mathcal{R}(P)$. By setting $N^c(1; P') = N^c(1; P_0) \cup \{l\} \setminus \{2\}$, we have $[3, s+2] \subset S(P')$ by (iv). Recall that $|S|$ is maximal, so $S(P') = [3, s+2]$ and $r' = r(P') = 3$. Again by Claim 3.3 (b), we know that

$$N^c(2; P') = \{3\} \cup [t', u'] \cup [w', l] \cup \{1\}$$

such that

$$c(2, j) = c(j, j+1) \neq c(j, j-1) \tag{9}$$

for $t' \leq j \leq u'$. Recall that $2 \in S(P_0)$, so $c(l, 2) = c(2, 3)$ by Lemma 2.2 (a). This means $l \notin N^c(2; P')$. Hence,

$$N^c(2; P') = \{3\} \cup [t', u'] \cup \{1\}.$$

Since $(2, 3, \dots, u', 2)$ is a p.c. cycle satisfying (1) for P' , $u' = d + 1$ or else the second assertion of Lemma 3.2 holds. Therefore, $N^c(2; P') = [3, d + 1] \cup \{1\}$ and so (v) holds by setting $N^c(2; P_0) = N^c(2; P')$. Note that $t' = 4$ and $u' = d + 1$, so (vi) is true by (9). In summary, we have shown that (i) – (vi) hold for all $i \geq 0$.

Recall that $d \geq 4$ and $s \geq 2$. By (iii) and (ii),

$$c(\phi^i(1), \phi^i(3)) \neq c(\phi^i(1), \phi^i(2)) = c(\phi^i(1), \phi^i(l)).$$

Similarly, we have

$$\begin{aligned} c(\phi^i(1), \phi^i(3)) &\neq c(\phi^i(3), \phi^i(2)) \\ c(\phi^i(2), \phi^i(4)) &\neq c(\phi^i(2), \phi^i(1)) \\ c(\phi^i(2), \phi^i(4)) &\neq c(\phi^i(4), \phi^i(4)) \end{aligned}$$

by (iv), (v) and (vi) respectively. In summary, we have

$$c(j, j + 2) \notin \{c(j - 1, j), c(j + 1, j + 2)\}$$

for $2 \leq j \leq l - 2$. Set $j = l - 2$ and recall that

$$c(1, 2) \neq c(1, l - 1) = c(l - 2, l - 1) \neq c(l - 1, l),$$

so $(1, 2, \dots, l - 2, l - 1, 1)$ is a p.c. cycles spanning $[l]$. This is a contradiction, so the claim is true. \square

Since $|S| \geq 2$, $t + 1 \in U$. By Claim 3.3 (b) and Claim 3.5, $(1, 2, \dots, t + 1, 1)$ is a cycle of length $t + 1 \geq r + 4 \geq 5$. Thus, we may assume that $d \geq 5$. Next we show that $|S|$ is at least three.

Claim 3.6. $|S| \geq 3$.

Proof of claim. Suppose the contrary, so $|S| = 2$ by Claim 3.4. Without loss of generality $r \neq 1$, otherwise consider the path $(2, 3, \dots, l, 1)$ instead. Thus,

$$A = [2, r] \cup \{t, t + 1\} \cup [w, l] \tag{10}$$

by (5). It should be noted that here t is not necessarily at least $r + 3$. We divide into separate cases depending on w .

Case 1: $w \leq l - 2$. Recall that $l \geq d + 2$ by the remark after (2). If $c(1, l - 1) \neq c(l - 1, l - 2)$, $(1, 2, \dots, l - 1, 1)$ is a p.c. cycle of length $l - 1 \geq d + 1$ and so Lemma 3.2 holds. Thus, $c(1, l - 1) = c(l - 1, l - 2)$. Note that both $l - 1$ and l are members of $S((l, l - 1, \dots, 1))$, so $|S((l, l - 1, \dots, 1))| \geq 2$. Since $|S| = 2$, $c(1, l - 2) \neq c(l - 2, l - 1)$ by Lemma 2.2 (a) taking $P = (l, l - 1, \dots, 1)$. In addition, $(1, 2, \dots, l - 2, 1)$ is a p.c. cycle of length at most d , so $l \leq d + 2$. Recall that $l \geq d + 2$, so $l = d + 2$. Moreover, $B = [2, d + 1]$ as $1, d + 2 \notin B$ and so $r = 2$ and $s = 3$. Since the p.c. cycle $C = (1, 2, 3, d + 2, d + 1, \dots, w, 1)$ has length d by Claim 3.3, we deduce that $w = 6$. As $3, 4 \in B$ and $3 \in S$, $c(4, d + 2) \neq c(3, d + 2) = c(3, 4)$. However, $(1, 2, 3, 4, d + 2, d + 1, \dots, 6, 1)$ is a p.c. cycle of length $d + 1$ which is a contradiction.

Case 2: $w = l - 1$. By (10),

$$A = [2, r] \cup \{t, t + 1, l - 1, l\} \tag{11}$$

and $d = |A| = r + 3$. Since $l \geq d + 2$, $c(1, l - 1) = c(l - 1, l - 2)$ or else $(1, 2, \dots, l - 1, 1)$ is a p.c. cycle of length $l - 1 \geq d + 1$. Note that both $l - 1$ and l are members of

$S((l, l-1, \dots, 1))$, so $|S((l, l-1, \dots, 1))| \geq 2$. By Claim 3.3 taking $P' = (l, l-1, \dots, 1)$, it is easy to deduce that $2 \in B$ and so $r = 2$ and $s = 3$. More importantly, we have $d = 5$ by (11). By considering the p.c. cycle $(1, 2, \dots, t+1, 1)$, we have either $t = 3$ or $t = 4$.

If $t = 3$, then $c(3, 4) = c(3, l)$ as $3 \in S$. Note that $c(1, 3) = c(3, 4)$ and $c(1, 4) = c(4, 5)$ by Claim 3.3 (b). Since $r = 2$, the path $P' = (3, 4, \dots, l, 2, 1)$ is a member of $\mathcal{R}(P)$. Observe that $S(P') = \{3, 4\}$ by setting $3, 4 \in N^c(1; P')$, so Claim 3.3 (b) implies that $N^c(3; P') = \{4\} \cup \{t_3, t_3 + 1\} \cup \{l, 2\}$ for some t_3 as $3 \in N^c(1; P')$. In particular, $4, l \in N^c(3, P')$ but $c(3, 4) = c(3, l)$, which is a contradiction.

If $t = 4$, then $A = \{2, 4, 5, l-1, l\}$ by (11) as $r = 2$. If $l = 7$, then $A = \{2, 4, 5, 6, 7\}$. Recall that $c(1, l-1) = c(l-1, l-2)$, so $c(1, 6) = c(5, 6)$. Note that $u = 5$, so by Lemma 2.2 (d) $c(1, 5) = c(5, 6) = c(1, 6)$. This is a contradiction as $5, 6 \in N^c(1)$, so $l \geq 8$. We now mimic the proof of Claim 3.5. Define ϕ to be the permutation on $[1, l]$ such that $(\phi(1), \phi(2), \dots, \phi(l)) = (3, 4, \dots, l, 2, 1)$. By a similar argument as in the proof of Claim 3.5, the following are true for $0 \leq i \leq (l-1)/2$:

- (i) $P_i = (\phi^i(1), \phi^i(2), \dots, \phi^i(l)) \in \mathcal{R}(P)$,
- (ii) $S(P_i) = \{\phi^i(j) : 1 \leq j \leq s\}$.
- (iii) $A_i = \{\phi^i(j) : j \in A\} = N^c(\phi^i(1); P_i)$,
- (iv) for $4 \leq j \leq 5$,

$$c(\phi^i(1), \phi^i(j)) = c(\phi^i(j), \phi^i(j+1)) \neq c(\phi^i(j), \phi^i(j-1)).$$

Hence by (iii) and (iv),

$$c(i, j) \notin \{c(i, i-1), c(j, j-1)\}$$

for $i \leq l-3$ odd and $j \in \{i+3, i+4\}$. If l is even, then, by taking $i = l-3$ and $j = l$, $(1, 2, \dots, l-3, l, l-1, 1)$ is a p.c. cycle of length $l-1 \geq 7$. If l is odd, then, by taking $i = l-4$ and $j = l$, $(1, 2, \dots, l-4, l, l-1, 1)$ is a p.c. cycle of length $l-2 \geq 6$. Both cases contradict our assumption of G and P .

Case 3: $w = l$. By (10), $A = [2, d] \cup \{l\}$, so $c(1, 2) \neq c(1, l) = c(l, l-1)$. In particular, $u = d$, $t = d-1$, $r = d-2$ and $s = d-1$, so

$$B \subset [d-2, l-1] \tag{12}$$

and $S = \{d-2, d-1\}$. Let $b \in B \cap [d, l-2]$. By the definition of B , $c(l, b) \neq c(l, l-1) = c(1, l)$. If $c(l, b) \neq c(b-1, b)$, then $(1, 2, \dots, b, l, 1)$ is a p.c. cycle of length $b+1 \geq d+1$, a contradiction. Thus,

$$c(l, b) = c(b-1, b) \text{ for } b \in B \cap [d, l-2]. \tag{13}$$

If $b = d$, then $c(l, d-1) = c(d-1, d) = c(l, d)$ as $d-1 \in S$, which contradicts the fact that $d-1, d \in B = N^c(l)$. If $b = d+1$, then $(1, d, d+1, l)$ is a monochromatic path of length 3. Let G' be the edge-coloured subgraph of G obtained by remove the edge $(d, d+1)$. Note that $\delta^c(G') = d$. The p.c. path

$$P' = (d, 1, 2, \dots, d-1, l, l-1, \dots, d+1)$$

can be obtained by a rotation of P with pivot point $d-1$ and endpoint 1 followed by a rotation with pivot point d and endpoint d . Hence, P' is maximal in G and also in G' and so G' contradicts the edge-minimality of G . Therefore, $B \cap \{d, d+1\} = \emptyset$, so (12) becomes

$$B \subset \{d-2, d-1\} \cup [d+2, l-1]. \tag{14}$$

Recall that $l \in A$, so the path $P_0 = (l-1, l-2, \dots, 1, l) \in \mathcal{R}(P)$. By (13), $B \cap [d, l-2] \subset S(P_0)$. Therefore, (14) implies that

$$\begin{aligned} 5 \leq d \leq |B| &= |\{d-2, d-1, l-1\} \cup (B \cap [d+2, l-2])| \\ &= 3 + |S(P_0)| \leq 5, \end{aligned}$$

so $d = 5$ and $r = 3$. If we replace P with P_0 and repeat all the arguments in the proof of this claim, then we can deduce that $B = \{3, 4, l-4, l-3, l-1\}$. By (14), $7 = d+2 \leq l-4$ and so $l \geq 11$, which implies Lemma 3.2. This completes the proof of the claim. \square

Next, we are going to show that $(r+1, r+3)$ is an edge such that

$$c(r+3, r+4) \neq c(r+1, r+3) \neq c(r+1, r+2). \quad (15)$$

By Claim 3.5, we may assume that $t \geq r+3$. First, let $P_1 = (r+1, r+2, \dots, l, r, \dots, 1) = (\phi_1(1), \phi_1(2), \dots, \phi_1(l))$. By Claim 3.6, $P_1 \in \mathcal{R}(P)$. It is easy to see that $S(P_1) = [t, u]$ and P_1 has a crossing by choosing $N^c(r+1; P_1)$ with $r \in N^c(r+1; P_1)$ and $N^c(1; P_1) = A$ if $r \neq 1$ and $N^c(1; P_1) = A \cup \{l\} \setminus \{2\}$ if $r = 1$. Since $r+1 \notin A$ as $t \geq r+3$, by Claim 3.3 (b) we have

$$N^c(r+1; P_1) = \{\phi_1(j) : j \in [2, r_1] \cup [t_1, u_1] \cup [w_1, l]\}$$

for some $2 \leq r_1 < t_1 < u_1 < w_1 \leq l$. Note that $\phi_1(r_1) = t \geq r+3$, so $r+3 \in N^c(r+1; P_1)$. Since $\{r, r+2, r+3\} \subset N^c(r+1; P_1)$, we can pick $N^c(r+1; P)$ such that $r+3 \in N^c(r+1; P)$. By Claim 3.3 and Claim 3.6, $r+2 \in S$ and so

$$P_2 = (r+3, r+4, \dots, l, r+2, \dots, 1) = (\phi_2(1), \phi_2(2), \dots, \phi_2(l)) \in \mathcal{R}(P).$$

By a similar argument, $S(P_2) = [t, u]$ and P_2 has a crossing. Therefore, $N^c(r+3; P_2) = \{\phi_2(j) : j \in A_2\}$, where

$$A_2 = \begin{cases} [2, r_2] \cup [t_2, u_2] \cup [w_2, l] & \text{if } t \neq r+3, \text{ or} \\ \{2\} \cup [t_2, u_2] \cup [w_2, l-1] & \text{if } t = r+3, \end{cases} \quad (16)$$

for some $2 \leq r_2 < t_2 < u_2 < w_2 \leq l$ by (5). Since $r+2 \in S$, $c(l, r+2) \neq c(r+2, r+3)$ by Lemma 2.2 (a). Thus, we may assume that $r+2 \in N^c(r+3; P_2) = \{\phi_2(j) : j \in [w_2, l-1]\}$, so

$$[2, r+2] \subset \{\phi_2(j) : j \in [w_2, l-1]\}.$$

In particular, $\{r+1, r+2, r+4\} \subset N^c(r+3; P_2)$, so we may assume that $r+1 \in N^c(r+3; P)$. In summary, we have shown that $r+3 \in N^c(r+1; P)$ and $r+1 \in N^c(r+3; P)$, so (15) holds.

Recall that $C = (1, 2, \dots, s, l, l-1, \dots, w, 1)$ is a p.c. cycle and $|C| = |[1, s]| + |[w, l]| = d$ by Claim 3.3 (b). If $s+1 \in B$, then $C' = (1, 2, \dots, s+1, l, l-1, \dots, w, 1)$ is a p.c. cycle of length $d+1$ satisfying (1), because $c(l, s+1) \neq c(l, s) = c(s, s+1)$ by Lemma 2.2 (b). Hence, $s+1 \notin B$. Therefore, by (15), $C' = (r+1, r+2, l, l-1, \dots, r+3, r+1)$, see Figure 4, is a p.c. cycle containing $B \cup \{l, s+1\} \setminus \{r\}$. Moreover, C' satisfies (1), a contradiction as $|C'| \geq d+1$. The proof of the Lemma 3.2 is completed. \square

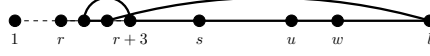


Figure 4: Cycle $(r+1, r+2, l, l-1, \dots, r+3, r+1)$

4 Graphs with short p.c. cycles

This section concerns graphs such that no p.c. cycle has length more than some fixed k . First we prove Proposition 1.5.

Proof of Proposition 1.5. Fixed k and we proceed by induction on d . For $d = k - 1$, Proposition 1.5 holds by considering $\tilde{G}(d; p)$ for $p \geq d$ as defined in Example 1.3. Thus, we may assume $d > k - l$. Let $\mathcal{G}(d - 1, k)$ be the family of edge-coloured graphs G with $\delta^c(G) \geq d - 1$ such that the longest p.c. paths and p.c. cycles in G are of lengths $k2^{d-k+1} - 2$ and $k - 1$ respectively. Note that $\mathcal{G}(d - 1, k)$ exists by induction hypothesis. We take $p \geq d$ vertex disjoint copies of members of $\mathcal{G}(d - 1, k)$, H_1, \dots, H_p . Take a new vertex x and add an edge of new colour c_j between x and every vertex of H_j for each $j \in [p]$. Call the resulting graph G' . It is easy to see by induction on d that every vertex in G' has colour degree at least d . Moreover, the longest p.c. path and p.c. cycle in $\tilde{G}(d, k)$ have lengths $k2^{d-k+2} - 2$ and $k - 1$ respectively. \square

We are going to prove Theorem 1.4 in the remaining of this section. First, we would need the following definitions. Let c be an edge colouring of a graph G such that $\delta^c(G) = d \geq 3$. Let $P = (1, 2, \dots, l)$ be a p.c. path in G . Define $f_i(P)$ to be the resultant path after a rotation of P pivoting at the i th element with the last vertex as the fixed endpoint. Similarly, define $g_j(P)$ to be the resultant path after a rotation of P pivoting at the j th element with the first vertex as the fixed endpoint. Thus, we consider f_i and g_j as permutations on P . For example, $f_3 \circ g_1(1, 2, 3, 4, 5, 6) = f_3(1, 6, 5, 4, 3, 2) = (6, 1, 5, 4, 3, 2)$. Furthermore, we only consider $f_i(P)$ and $g_j(P)$ if $f_i(P)$ and $g_j(P)$ are p.c. paths respectively. In other words, if $c(1, 2) \neq c(1, i) \neq c(i, i+1)$, then $f_i(P)$ is defined, and a similar statement for $g_j(P)$. Let $\mathcal{R}'(P)$ is the set of p.c. paths that can be obtained by a sequence of rotations of P . Note that $\mathcal{R}(P) = \{P', h(P') : P' \in \mathcal{R}'(P)\}$, where $h(P')$ is a reflection of P' . We study some basic properties of $\mathcal{R}'(P)$ in the coming proposition.

Proposition 4.1. *Let c be an edge colouring of a graph G such that $\delta^c(G) = d \geq 3$. Let P be a properly coloured path in G . Then the following statements holds:*

- (a) *If $f_i(P') \in \mathcal{R}'(P)$, then $P' \in \mathcal{R}'(P)$.*
- (b) *If $g_j(P') \in \mathcal{R}'(P)$, then $P' \in \mathcal{R}'(P)$.*
- (c) *For $i \leq j$, f_i and g_j commutes.*

Furthermore, suppose that P' has no crossing for all $P' \in \mathcal{R}'(P)$. Then

- (d) *every $P' \in \mathcal{R}'(P)$ can be obtained from P by a sequence of f_{i_1}, \dots, f_{i_a} followed by a sequence of g_{j_1}, \dots, g_{j_b} and visa versa.*
- (e) *there exist integers $i_0 \leq j_0$ depending only on $\mathcal{R}'(P)$ such that $i_{a'} \leq i_0$ and $j_0 \leq j_{b'}$ for $a' \in [a]$ and $b' \in [b]$.*

Proof. Let $P' = (1, 2, \dots, l)$ and so $f_i(P') = (i-1, i-2, \dots, 1, i, i+1, \dots, l)$. Since P' is a p.c. path, we must have $c(i-1, i-2) \neq c(i-1, i) \neq c(i, i+1)$. Thus, $P' = f_i \circ f_i(P') \in \mathcal{R}'(P)$ and so (a) holds. By a similar argument, (b) holds. Note that f_i (and g_j) reverses the

ordering in the first $(i - 1)$ elements (and the last $(l - j)$ elements respectively). Hence, (c) follows easily.

Assume that P' has no crossing for all $P' \in \mathcal{R}'(P)$. In order to prove (c), it is enough to show that if $f_i \circ g_j(P') \in \mathcal{R}'(P)$, then $f_i \circ g_j(P') = g_j \circ f_i(P')$. By (a) and (b), we have $g_j(P'), P' \in \mathcal{R}'(P)$. Recall that $P' = (1, 2, \dots, l)$, so $g_j(P') = (1, 2, \dots, j, l-1, \dots, j+1)$. Note that $i \in N^c(1; g_j(P'))$ for some $N^c(1; g_j(P'))$ as $f_i(g_j(P'))$ is defined. Fix one such $N^c(1; g_j(P'))$. Since P is p.c., we may pick $N^c(j+1; g_j(P'))$ such that $j \in N^c(j+1; g_j(P'))$. Recall that $g_j(P')$ has no crossing, so

$$i \leq \max\{i' \in N^c(1; g_j(P'))\} \leq \min\{j' \in N^c(j+1; g_j(P'))\} \leq j. \quad (17)$$

By (c), $f_i \circ g_j(P') = g_j \circ f_i(P')$. Let i_0 (and j_0) be the maximal integer i (and the minimal integer j) such that $i \in N^c(i_1; P'')$ (and $j \in N^c(i_l; P'')$) for $P'' = (i_1, \dots, i_l) \in \mathcal{R}'(P)$. Thus, (e) follows from (d) and (17). \square

Let G be an edge-colouring graph such that no p.c. cycle has length more than some fixed k . The next lemma show that the length of every maximal p.c. path grows exponentially in $\delta^c(G)$. Thus, Lemma 4.2 trivially implies Theorem 1.4. The main idea of the proof of the lemma is as follows. Let $P = (1, 2, \dots, l)$ be a maximal p.c. path in G . By Lemma 3.1 and Lemma 3.2, P does not have a crossing. Our aim is to find integers $1 < x < y < l$ such that $P_x = (1, 2, \dots, x)$ and $P_y = (y+1, y+2, \dots, l)$ are maximal p.c. path in $G_x = G \setminus \{x+1\}$ and $G_y = G \setminus \{y-1\}$ respectively. Clearly, $\delta^c(G_x) \geq \delta^c(G) - 1$. By inducting on $\delta^c(G)$, we can deduce that P_x is very long (exponentially in $\delta^c(G_x)$), and a similar statement holds for P_y . Thus, P is also very long. We would like to point out that the condition $d \geq \lceil 3k/2 \rceil - 3$ is an artifact of our proof.

Lemma 4.2. *Let $k \geq 3$ and $d \geq \lceil 3k/2 \rceil - 3$ be integers. Let c be an edge colouring of a graph G such that $\delta^c(G) = d$. Suppose G does not contain any p.c. cycle of length at least k . Then, every maximal p.c. path in G has length at least $k2^{d-\lceil 3k/2 \rceil+4} - 2$.*

Proof. Let $P = (1, 2, \dots, l)$ be a maximal p.c. path in G . We are going to show that $l \geq k2^{d-\lceil 3k/2 \rceil+4} - 1$ by induction on d . If $d = \lceil 3k/2 \rceil - 3 \geq k - 1$, then P does not have a crossing. Otherwise Lemma 3.1 and Lemma 3.2 implies that G contains a p.c. cycle of length at least k or $l \geq 2d + 1 \geq 2k - 1$ as required. Since P does not have a crossing, we have

$$l \geq |(N^c(1) \cup \{1\}) \cup (N^c(l) \cup \{l\})| \geq 2(d+1) - 1 \geq 2k - 1.$$

Thus, the lemma is true for $d = \lceil 3k/2 \rceil - 3$. Hence, we may assume that $d \geq \lceil 3k/2 \rceil - 2 \geq k$. If P has a crossing, then by Lemma 3.2 there is a p.c. cycle of length at least $d \geq \lceil 3k/2 \rceil - 2 \geq k$, which is a contradiction. Thus, no $P' \in \mathcal{R}'(P)$ has a crossing.

Define $X = X(P)$ to be the set of all possible i_1 such that there exists a path $P' = (i_1, \dots, i_l) \in \mathcal{R}'(P)$. Similarly, define $Y = Y(P)$ to be the set of all possible i_l . Clearly, $1 \in X$ and $l \in Y$. Let $x = \max\{i \in X\}$ and $y = \min\{j \in Y\}$. If $f_i(P') \in \mathcal{R}'(P)$, then $i \leq x + 1$. If $g_j(P') \in \mathcal{R}'(P)$, then $j \geq y - 1$. By Proposition 4.1 (e), $x < y$. Since P is maximal, $N^c(1; P) \subset [l]$. If $i' \in N^c(1; P)$ is maximal, then $c(1, i') = c(i', i' - 1)$ or else $(1, 2, \dots, i', 1)$ is a cycle of length at least $d + 1$. Hence, $i' - 1 \in X$ as $f_{i'}(P) \in \mathcal{R}'(P)$. Thus, equivalently

$$x = \max\{i - 1 : i \in N^c(i_1; P') \text{ for } P' = (i_1, \dots, i_l) \in \mathcal{R}'(P)\}. \quad (18)$$

Without loss of generality, we may assume that $x+1 \in N^c(1; P)$. By a similar argument, if $j' \in N^c(l; P)$ is minimal, then $j'+1 \in Y$. By Proposition 4.1 (d) and (e), we may further assume that $y-1 \in N^c(l; P)$.

Let P_x and P_y be the p.c. paths $(1, 2, \dots, x)$ and $(y, y+1, \dots, l)$ respectively. Let G_x and G_y be the graphs $G - \{x+1\}$ and $G - \{y-1\}$ respectively. Clearly, $\delta^c(G_x), \delta^c(G_y) \geq d-1$. Suppose that P_x and P_y are maximal in G_x and G_y respectively. By the induction hypothesis, $|P_x|, |P_y| \geq k2^{d-\lceil 3k/2 \rceil+3} - 1$. Note that $x+1$ is not a vertex in P_x nor P_y , so

$$l \geq |P_x| + |P_y| + 1 \geq 2(k2^{d-\lceil 3k/2 \rceil+3} - 1) + 1 = k2^{d-\lceil 3k/2 \rceil+4} - 1$$

as required. Hence, in proving the lemma, it suffices (by symmetry) to show that P_x is maximal in G_x .

Claim 4.3. (a) If $P' = (i_1, i_2, \dots, i_x) = f_{j_a} \circ \dots \circ f_{j_1}(P_x)$, then $c(i_x, i_{x-1}) = c(x, x-1)$.

Moreover, $(i_1, i_2, \dots, i_x, x+1, x+2, \dots, l) \in \mathcal{R}'(P)$

(b) If $P' = (i_1, i_2, \dots, i_x) = g_{j_b} \circ \dots \circ g_{j_1}(P_x)$, then $c(i_1, i_2) = c(1, 2)$. Moreover, $(i_x, i_{x-1}, \dots, i_1, x+1, x+2, \dots, l) \in \mathcal{R}'(P)$.

Proof. Note that f_j fixed the last two elements unless $j = x$. Also, $i_x = x$. Thus, in proving the first assertion of (a) it is enough to consider the case when $a = 1$ and $j_1 = x$. Since $N^c(1; P) \subset P_x \cup \{x+1\}$, $|P_x| \geq d$. If $x \in N^c(1; P)$, then $c(1, x) = c(x, x-1)$ or else $(1, 2, \dots, x, 1)$ is a p.c. cycle of length at least $d-1 \geq k$. Observe that $P'_x = f_x(P_x) = (x-1, x-2, \dots, 1, x)$ and so the first assertion of (a) holds. Recall that $c(x-1, x) \neq c(x, x+1)$ as P is a p.c. path, so the second assertion follows.

By a similar argument, first assertion of (b) also holds. Recall that $c(1, 2) \neq c(1, x+1) = c(x, x+1) \neq c(x+1, x+2)$, so the second assertion of (b) also holds. \square

First suppose that no $P' \in \mathcal{R}'(P_x)$ has a crossing. Let $X_x = X(P_x)$ and $Y_x = Y(P_x)$. If $P'_x = (i_1, \dots, i_x) \in \mathcal{R}'(P_x)$ is extensible in G_x , then there exists $u \notin [x+1]$ such that (i_1, \dots, i_x, u) or (u, i_1, \dots, i_x) is a p.c. path in G_x . Assume that (i_1, \dots, i_x, u) is a p.c. path in G_x . Note that $c(i_x, u) \neq c(i_x, i_{x-1})$. By Proposition 4.1 (d), $P'_x = f_{i_a} \circ \dots \circ f_{i_1} \circ g_{j_b} \circ \dots \circ g_{j_1}(P_x)$. By Proposition 4.1 (a), we may assume without loss of generality that $P'_x = g_{j_b} \circ \dots \circ g_{j_1}(P_x)$ as the statement $c(i_x, u) \neq c(i_x, i_{x-1})$ still holds by Claim 4.3 (a). By Proposition 4.1 (b), we have $P' = (i_x, i_{x-1}, \dots, i_1, x+1, x+2, \dots, l) \in \mathcal{R}(P)$. Pick $N^c(i_x; P')$ such that $u \in N^c(i_x; P')$. If $u \in [l]$, then $u > x+1$ contradicting (18). If $u \notin [l]$, then P' is extensible contradicting the maximality of P . A similar argument also holds if (u, i_1, \dots, i_x) is a p.c. path in G_x . Thus, $P'_x \in \mathcal{R}'(P)$ is not extensible in G_x and so P_x is maximal in G_x .

Now suppose there exists $P' = (i_1, \dots, i_x) \in \mathcal{R}'(P_x)$ that has a crossing. The next claim allows us to assume without loss of generality that $i_1, i_x \in X$.

Claim 4.4. There exists $P' = (i_1, \dots, i_x) \in \mathcal{R}'(P_x)$ that has a crossing such that $i_1, i_x \in X$. Moreover, $N^c(i_1; P') \cup N^c(i_x; P') \subset [x+1]$. Furthermore, if $x+1 \in N^c(i_1, P')$, then $(i_x, i_{x-1}, \dots, i_1, x+1, x+2, l) \in \mathcal{R}'(P)$. Similarly, if $x+1 \in N^c(i_x, P')$, then $(i_1, i_2, \dots, i_x, x+1, x+2, l) \in \mathcal{R}'(P)$.

Proof of claim. Let P' be obtained from P_x by a combinations of rotations $f_{j_1}, \dots, f_{j_a}, g_{j'_1}, \dots, g_{j'_b}$ with multiplicities. We further assume that P' requires the smallest number of rotations. Recall by Proposition 4.1 (c) that if $j_{a'} \leq j'_{b'}$ for $a' \in [a]$ and $b' \in [b]$, then $f_{j_{a'}}$ and

$g_{j_{b'}}'$ commutes. Since P' uses the smallest number of rotations, $j_{a'} \leq j_{b'}$ for $a' \in [a]$ and $b' \in [b]$. Thus,

$$P' = g_{j_b'} \circ \cdots \circ g_{j_1'} \circ f_{j_a} \circ \cdots \circ f_{j_1}(P_x).$$

By Proposition 4.1 (a),

$$P'' = (i_1 = i'_1, i'_2, \dots, i'_x) = f_{j_a} \circ \cdots \circ f_{j_1}(P_x) \in \mathcal{R}'(P_x)$$

Therefore, $i_1 \in X$ by Claim 4.3 (a) as

$$\tilde{P} = (i'_1, i'_2, \dots, i'_x, x+1, x+2, \dots, l) \in \mathcal{R}'(P).$$

Similarly, $i_x \in X$. By Proposition 4.1 (b), we know that the edges that are incident with i_1 in P' and P'' have the same colours. Thus, each choice of $N^c(i_1; P')$ naturally induces $N^c(i_1; P'')$. Set $N^c(i_1; \tilde{P}) = N^c(i_1; P'')$. In addition, $N^c(i_1; \tilde{P}) \subset [x+1]$ as P is maximal and (18), so $N^c(i_1; P') \subset [x+1]$ and similarly $N^c(i_x; P') \subset [x+1]$.

Now suppose that $x+1 \in N^c(i_1, P')$ and so $x+1 \in N^c(i_1, \tilde{P})$. Observe that f_{x+1} reserve the ordering of the first x elements in \tilde{P} , so we can view it as a reflection on P'' . Therefore,

$$(i_x, i_{x-1}, \dots, i_1, x+1, x+2, l) = f_{x-j_b'+1} \circ \cdots \circ f_{x-j_1'+1} \circ f_x(\tilde{P})$$

is a member of $\mathcal{R}'(P)$ as required. By a similar argument, the finally assertion also holds if $x+1 \in N^c(i_x; P')$. Hence, the proof of the claim is completed. \square

For convenience, we abuse the notation and assume that $P' = (1, 2, \dots, x)$, so 1 and x are not necessarily adjacent to $x+1$. We now mimic the proof of Lemma 3.2 on P' . Let $A = N^c(1; P') \setminus \{x+1\}$ and $B = N^c(x; P') \setminus \{x+1\}$. Note that

$$1 \notin B \text{ or } x \notin A. \quad (19)$$

Otherwise, $(1, 2, \dots, x, 1)$ is a p.c. cycle of length $x \geq d \geq k$. Let $r = \min\{b \in B\}$. Recall that $\delta^c(G_x) \geq d-1$ and P' is not extensible in G_x as $A \cup B \subset [x]$. By taking $G = G_x$ and $P = P'$ in Lemma 2.2, we can find $s \in S$ satisfying Lemma 2.2 (a) – (c). Let $S = [r, s] \cap B$. If $b \in B$ and $b \leq x-k+1$, then $c(x, b) = c(b, b+1)$ or else (b, \dots, x, b) is a p.c. cycle of length at least k . Thus,

$$|S| \geq |B| - k + 2. \quad (20)$$

Since $|S| \geq 2$, $s \geq 2$ and so we can also find $u, w \in A$ satisfying Lemma 2.2 (d) – (f). It should be noted that here $|S|$ is not assumed to be maximal over all such P' . Let

$$s' = \min\{a \in A : c(1, j) = c(j, j-1) \text{ for } j \in A \cap [a, x]\}$$

and $S' = A \cap [s', x]$. By Lemma 2.2 (d) and (e), $w \leq s'$. If $i \in A$ and $i \geq k$, then $c(1, i) = c(i, i-1)$ or else $(1, 2, \dots, i, 1)$ is a p.c. cycle of length at least k . Therefore,

$$|S'| \geq |A| - k + 2. \quad (21)$$

By Lemma 2.2 (a) and (e) and Lemma 2.1,

$$C = (1, 2, \dots, s, x, x-1, \dots, w, 1)$$

is a p.c. cycle. By our assumption in the hypothesis of Lemma 4.2, we know that $|C| \leq k - 1$. For the remaining of the proof, our aim is to show that this would lead to a contradiction.

Note that $|\{1, x\} \setminus (A \cup B)| \geq 1$ by (19). If $x+1 \notin N^c(1; P') \cup N^c(x; P')$, then $|A|, |B| \geq d$ and so by (20) and (21)

$$\begin{aligned} k-1 &\geq |C| = |[1, s] \cup [w, x]| \\ &\geq |S| + |S'| + |\{1, x\} \setminus (S \cup S')| \\ &\geq 2(d - k + 2) + |\{1, x\} \setminus (A \cup B)| \geq 2d - 2k + 5 \\ 3k/2 - 3 &\geq d, \end{aligned}$$

which is a contradiction as d is assumed to be at least $\lceil 3k/2 \rceil - 2$. Without loss of generality, we may assume that $x+1 \in A$. By Claim 4.4,

$$(x, x-1, \dots, 1, x+1, x+2, \dots, l) \in \mathcal{R}'(P). \quad (22)$$

Let $\mathcal{R}_g(P')$ be the sets of p.c. path obtained from a sequences of g_j 's. By Claim 4.3 (b), we may assume that $x+1 \in N^c(i_1; P'')$ for all $P'' \in \mathcal{R}_g(P')$ and so the other endpoint of P'' is also a member of X by (22). From now on, we further assume that $|S'| = |S'(P')| \geq |S'(P'')|$ for $P'' \in \mathcal{R}_g(P')$.

We now give a lower bound on $|S'|$. By Lemma 2.2 (a), the path $P'' = g_r(P') = (1, 2, \dots, r, x, x-1, \dots, r+1) \in \mathcal{R}_g(P')$ and so $r+1 \in X$ and $x+1 \in N^c(1; P'')$. For $a \in A \cap [r+1, u] \setminus \{2\}$, $c(1, a) = c(a, a+1) \neq c(a, a-1)$ by Lemma 2.2 (c) and (d). Thus, $S'(P'') \supset A \cap [r+1, u] \setminus \{2\}$. Since $|S'|$ is maximal, we have

$$|S'| \geq |A \cap [r+1, u]| - \delta_{1,r}, \quad (23)$$

where $\delta_{1,r} = 1$ if $r = 1$ and $\delta_{1,r} = 0$ otherwise. Recall that the cycle $C = (1, 2, \dots, s, x, x-1, \dots, w, 1)$ has length at most $k-1$ and $1 \notin A$, so

$$\begin{aligned} k-1 &\geq |C| = |A| + |[1, s] \setminus A| + |[w, x] \setminus A| - |[s+1, u] \cap A| \\ &\geq d + |[r+1, s] \setminus A| + |\{x\} \setminus A| - |[s+1, u] \cap A| \\ |[s+1, u] \cap A| &\geq d - k + 1 + |[r+1, s] \setminus A| + |\{x\} \setminus A|. \end{aligned} \quad (24)$$

Recall that $x \notin A$ or $1 \notin B$ by (19). Thus, (23) becomes

$$\begin{aligned} |S'| &\geq d - k + 1 + |\{x\} \setminus A| + |[r+1, s]| - \delta_{1,r} \\ &\geq d - k + 1 + |[r+1, s]| \geq d - k + |S|. \end{aligned}$$

Recall by (20) that

$$|S| \geq |B| - k + 2 \geq d - k + 1$$

and $|\{1, x\} \setminus (A \cup B)| \geq 1$ by (19). Since $S \subset [1, s]$, $S' \subset [w, l]$ and $s < w$,

$$\begin{aligned} k-1 &\geq |C| = |[1, s] \cup [w, l]| \\ &\geq |S| + |S'| + |\{1, x\} \setminus (S \cup S')| \\ &\geq 2|S| + d - k + 1 \geq 3(d - k + 1), \end{aligned}$$

which is a contradiction as $d \geq \lceil 3k/2 \rceil - 2$. This completes the proof of Lemma 4.2. \square

5 The longest p.c. path

In this section, we prove Theorem 1.10. A directed graph H with base graph G is a order paired $(V(G), A(H))$ such that the ordered pair (u, v) or (v, u) is in $A(H)$ for $uv \in E(G)$.

Proof of Theorem 1.10. Let c be an edge colouring of a graph G such that $\delta^c(G) = d$. Let $P = (1, 2, \dots, l)$ be a maximal p.c. path in G , $N^c(1; P), N^c(l; P) \subset [l]$. Assume that $l < 6d/5$, or else there is nothing to prove. By greedy algorithm, $l \geq d + 1$, so $d \geq 6$. We may further assume that there is no p.c. cycle C spanning $[l]$. Otherwise, C is a p.c. Hamiltonian cycle if $|G| = l$ or we can find a p.c. path of length l by connectedness of G if $l < |G|$. Since $l < 6d/5$, P has a crossing. By Lemma 3.2, there exist a p.c. cycle C and a p.c. path Q such that

- (i) $C = (i_1, i_2, \dots, i_p, i_1)$ with $p \geq d + 1$,
- (ii) $Q = (i'_1, i'_2, \dots, i'_q)$,
- (iii) $V(C) \cap V(Q) = \emptyset$ and $V(P) = V(C) \cup V(Q)$,
- (iv) there exists $j \in [p]$ with $(i'_1, i_j) \in E(G)$ and $c(i'_1, i'_2) \neq c(i'_1, i_j)$.

Note that $p + q = l$ and $q \geq 1$. We may assume that q is minimal. Furthermore, by relabelling i_1, i_2, \dots, i_p if necessary,

$$P' = (i_1, i_2, \dots, i_p, i'_1, i'_2, \dots, i'_q)$$

is a p.c. path by (iv) and (i), so $N^c(i'_q; P') \subset V(P)$. Recall that $q = l - p \geq l - d - 1$ by (i). Hence,

$$|V(C) \cap N^c(i'_q; P')| \geq d - q + 1 = 2d - l + 2.$$

Define

$$R = \{i_j \in V(C) \cap N^c(i'_q; P') : c(i'_q, i_j) \neq c(i_j, i_{j+1})\}, \text{ and} \\ R' = \{i_{j-1} : i_j \in R\},$$

where we take i_0 to be i_p . By reversing the order of $\{i_1, i_2, \dots, i_p\}$ if necessary, we may assume that

$$|R'| = |R| \geq |V(C) \cap N^c(i'_q; P')|/2 \geq (2d - l + 2)/2. \quad (25)$$

For distinct $i_j, i_{j'} \in R'$, observe that

$$c(i_j, i_{j+1}) \neq c(i'_q, i_{j+1}) \neq c(i'_q, i_{j'+1}) \neq c(i_{j'}, i_{j'+1}).$$

If $(i_j, i_{j'})$ is an edge with $c(i_j, i_{j-1}) \neq c(i_j, i_{j'}) \neq c(i_{j'}, i_{j'-1})$, then $|j - j'| \geq 2$ and moreover

$$C' = (i'_q, i_{j+1}, i_{j+2}, \dots, i_{j'}, i_j, i_{j-1}, \dots, i_{j'+1}, i'_q)$$

is a p.c. cycle, see Figure 5. However, this contradicts the minimality of q by setting $C = C'$ and $Q = (i'_1, i'_2, \dots, i'_{q-1})$. Thus, if $(i_j, i_{j'})$ is an edge, then $c(i_j, i_{j'}) = c(i_j, i_{j-1})$ or $c(i_j, i_{j'}) = c(i_{j'}, i_{j'-1})$.

Define a directed graph H on R' such that there is an arc from i_j to $i_{j'}$ unless $(i_j, i_{j'})$ is an edge and $c(i_j, i_{j'}) \neq c(i_{j'}, i_{j'-1})$. Recall that if $i_j, i_{j'} \in R'$ and $(i_j, i_{j'})$ is an edge, then $c(i_j, i_{j'}) = c(i_j, i_{j-1})$ or $c(i_j, i_{j'}) = c(i_{j'}, i_{j'-1})$. Therefore, the base graph of H is

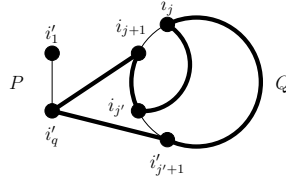


Figure 5: Cycle $(i'_q, i_{j+1}, i_{j+2}, \dots, i_{j'}, i_j, i_{j-1}, \dots, i_{j'+1}, i'_q)$

complete. Thus, there exists a vertex $i_{j_0} \in R'$ with in-degree at least $(|R'| - 1)/2$ in H . This means that

$$\begin{aligned} & |\{i_j \in R' \setminus \{i_{j_0}\} : (i_j, i_{j_0}) \notin E(G) \text{ or } c(i_j, i_{j_0}) = c(i_{j_0}, i_{j_0-1})\}| \\ & \geq (|R'| - 1)/2 \geq (2d - l)/4 \end{aligned} \quad (26)$$

by (25). Recall that $i_{j_0+1} \in R \subset N^c(i'_q; P')$, so $c(i'_q, i'_{q-1}) \neq c(i'_q, i_{j_0+1}) \neq c(i_{j_0+1}, i_{j_0+2})$. Note that $P'' = (i'_1, i'_2, \dots, i'_q, i_{j_0+1}, i_{j_0+2}, \dots, i_{j_0})$ is a p.c. path. Since P is a p.c. path of maximal length, each vertex $j \in V(G)$ such that (i_{j_0}, j) is an edge and $c(i_{j_0}, j) \neq c(i_{j_0}, i_{j_0-1})$ must be in $V(P'') = V(P) = [l]$. There are at least $d - 1$ such vertices as $\delta^c(G) = d$. Therefore, together with (26) we have

$$l \geq |\{i_{j_0}\}| + d - 1 + (2d - l)/4 = (6d - l)/4,$$

a contradiction as $l < 6d/5$. \square

6 k -edge colourings

In this section, we consider edge colouring with bounded number of colours. A k -edge colouring of a graph G uses k colours, c_1, c_2, \dots, c_k . Let G^{c_i} be the subgraph of G induced by edges of colour c_i . For a k -edge coloured graph G , define $\delta_k^{mon}(G) = \min\{\delta(G^{c_i}) : 1 \leq i \leq k\}$. Chapter 16 of [3] gives a good survey on $\delta_k^{mon}(G)$.

Aboulaoualim et al. [1] proved that if G is a 2-edge coloured graph with $\delta_2^{mon}(G) = \delta \geq 1$, then G has a p.c. path of length 2δ . They then conjectured that if G is a k -edge coloured graph with $k \geq 3$ and $\delta_k^{mon}(G) = \delta \geq 1$, then G has a p.c. path of length $\min\{|G| - 1, 2k\delta\}$. By modifying the construction of $\hat{G}(n, d)$ in Example 1.8, we show that this conjecture is false. Moreover, for every $\delta \geq 1$ and $k \geq 3$, there exists a k -edge coloured graph G such that $\delta_k^{mon}(G) \geq \delta$ and no p.c. path has length more than $\lfloor 3\delta(k + \epsilon)/2 \rfloor$, where $\epsilon = 1$ if k even, and $\epsilon = 0$ otherwise.

Proposition 6.1. *Let $k \geq 3$ and $\delta \geq 1$ be integers. Then there exists k -edge coloured graphs G of order $n \geq (k + \epsilon)\delta$ such that $\delta_k^{mon}(G) \geq \delta$ and no properly coloured path in G has length more than $\lfloor 3\delta(k + \epsilon)/2 \rfloor$, where $\epsilon = 1$ if k even, and $\epsilon = 0$ otherwise.*

Proof. Suppose $\delta = 1$. Partition $V(G)$ into X and Y with $X = \{x_1, x_2, \dots, x_{k+\epsilon}\}$. For k odd, let $G[X]$ be a complete graph. It is well known that $G[X]$ can be properly k -edge-coloured with colours $\{c_1, c_2, \dots, c_k\}$. Fix one such proper k -edge colouring of $G[X]$. Moreover, we may assume that x_i is not incident with an edge of colour c_i for $i \in [k]$. For k even, let H be a complete graph on X . By a similar argument, we can find a properly $(k + 1)$ -edge colouring with colours $\{c_1, c_2, \dots, c_{k+1}\}$ on H such that x_i is not incident

with an edge of colour c_i for $i \in [k+1]$. Observe that $H^{c_{k+1}}$, the induced subgraph of H by edges of colour c_{k+1} , is a perfect matching on the vertex set $\{x_1, \dots, x_k\}$. Let $G[X]$ be the properly k -edge-coloured graph $H - H^{c_{k+1}}$. In summary, $G[X]$ is a properly k -edge-coloured graph with colours $\{c_1, c_2, \dots, c_k\}$. Moreover, x_i is not incident with an edge of colour c_i for $i \in [k]$. Let $G[Y]$ be empty. For each $y \in Y$, add an edge of colour c_i between y and x_i for $i \in [k]$. By our construction, $\delta_k^{mon}(G) = 1$. Moreover, for each pair $y, y' \in Y$, every p.c. path from y to y' must contain at least two vertices in X . Thus, no path in G has length more than $\lfloor 3|X|/2 \rfloor$.

For $\delta \geq 2$, the proposition is proved by considering a δ -blow-up of G , that is, each vertex of G is replaced by δ independent vertices, and add an edge of colour c_i between each copy of v and each copy of u if and only if u and v are joined by an edge of colour c_i in G . \square

On the other hand, we show that if G is k -edge-coloured connected graph with $\delta_k^{mon}(G) \geq d$, then there exists a p.c. path of length $(10(k-1)\delta+8)/9-1$ or G contains a p.c. Hamiltonian cycle.

Theorem 6.2. *For integers $k \geq 2$, every k -edge-coloured connected graph G with $\delta_k^{mon}(G) \geq 1$ contains a p.c. path of length $(10(k-1)\delta_k^{mon}+8)/9-1$ or a p.c. Hamiltonian cycle.*

Proof. Let G be a connected graph with a k -edge colouring c such that $\delta_k^{mon}(G) \geq \delta$. Let $P = (1, 2, \dots, l)$ be a p.c. path in G of maximal length. We may assume the $l < (10(k-1)\delta+8)/9$ or else there is nothing to prove. We may further assume that no p.c. cycle spans the vertex set $[l]$, otherwise there exists a p.c. path of length l as G is connected, or G has a p.c. Hamiltonian cycle if $l = |G|$.

Define A be the set of vertices v such that $(1, v)$ is an edge and $c(1, v) \neq c(1, 2)$. Note that $|A| \geq (k-1)\delta$ and $A \subset [2, l]$ by the maximality of P . Similarly, define B be the set of vertices v such that (l, v) is an edge and $c(l, v) \neq c(l, l-1)$. By a similar argument, $|B| \geq (k-1)\delta$ and $B \subset [l-1]$. Let

$$I = \{i \in [l-1] : i+1 \in A \text{ and } i \in B\}.$$

Thus, $|I| \geq 2(k-1)\delta - l + 1$. If both 1 and $l-1$ are members of I , then $l \in A$ and $1 \in B$. This implies that $(1, 2, \dots, l, 1)$ is a p.c. cycle spanning $[l]$, a contradiction. So at most one of 1 and $l-1$ is a member of I and

$$|I \setminus \{1, l-1\}| \geq |I| - 1 \geq 2(k-1)\delta - l. \quad (27)$$

Let

$$I_1 = \{i \in I \cap [l-2] : c(1, i+1) = c(i+1, i+2)\} \subset [2, l-2].$$

Note that $1 \notin I_1$, because $c(1, 2) \neq c(2, 3)$. If $c(1, i+1) \neq c(i+1, i+2)$ and $c(l, i) \neq c(i, i-1)$ for $i \in I \setminus \{1, l-1\}$, then $(1, 2, \dots, i, l, l-1, \dots, i+1)$ is a cycle spanning $[l]$ as $i+1 \in A$ and $i \in B$. Thus, $c(1, i+1) = c(i+1, i+2)$ or $c(l, i) = c(i, i-1)$ for $i \in I \setminus \{1, l-1\}$. Hence, if $i \in I \setminus \{I_1 \cup \{1, l-1\}\}$, then $c(l, i) = c(i, i-1)$. Without loss of generality (by replace P with its reflection if necessary), we may assume by (27) that

$$|I_1| \geq |I \setminus \{1, l-1\}|/2 \geq (2(k-1)\delta - l)/2.$$

There exists $i_0 \in [l]$ such $|I_1 \cap [i_0]|, |I_1 \cap [i_0, l]| \geq |I_1|/2$. By removing at most one vertices from I_1 , we further assume that $|I_1 \cap [i_0]| = |I_1 \cap [i_0, l]| \geq |I_1|/2$. Let

$$\begin{aligned} R &= \{i+2 : i \in I_1 \cap [i_0, l]\}, \\ S &= \{i+1 : i \in I_1 \cap [i_0] \text{ and } c(i, l) \neq c(i, i-1)\}, \text{ and} \\ T &= \{i-1 : i \in I_1 \cap [i_0] \text{ and } c(i, l) = c(i, i-1)\}. \end{aligned}$$

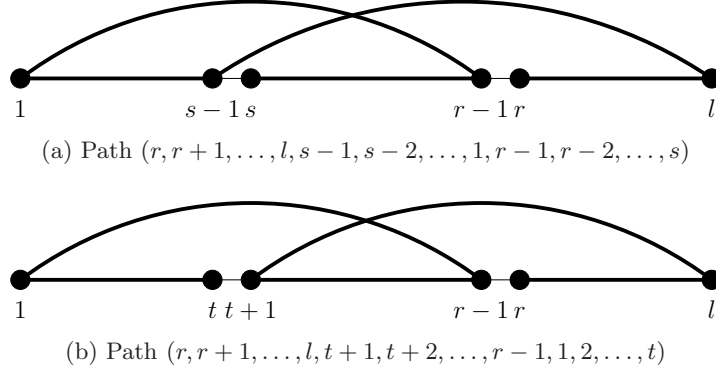


Figure 6

In summary, we have the followings:

- (a) for $r \in R$, $c(1, 2) \neq c(1, r-1) = c(r-1, r) \neq c(r-1, r-2)$,
- (b) for $s \in S$, $c(l, l-1) \neq c(l, s-1) \neq c(s-1, s-2)$,
- (c) for $t \in T$, $c(l, l-1) \neq c(l, t+1) = c(t+1, t) \neq c(t+1, t+2)$,
- (d) $\max\{u \in S \cup T\} < \min\{r \in R\}$,
- (e) $|R|, |S| + |T| \geq (2(k-1)\delta - l)/4$

Moreover, we can deduce that

- (f) $(r, r+1, \dots, l, s-1, s-2, \dots, 1, r-1, r-2, \dots, s)$ is a p.c. path for $r \in R$ and $s \in S$
- (g) $(r, r+1, \dots, l, t+1, t+2, \dots, r-1, 1, 2, \dots, t)$ is a p.c. path for $r \in R$ and $t \in T$.

See Figure 6 (a) and (b)

Define a directed bipartite graph H on vertex classes R and $S \cup T$ such that for $r \in R$, $s \in S$, $t \in T$ and $u \in S \cup T$,

- there is an arc from u to r unless $(r, u) \in E(G)$ and $c(r, u) \neq c(r, r+1)$,
- there is an arc from r to s unless $(r, s) \in E(G)$ and $c(r, s) \neq c(s, s-1)$,
- there is an arc from r to t unless $(r, t) \in E(G)$ and $c(r, t) \neq c(t, t+1)$.

Suppose that H has maximal in-degree $\Delta_-(H)$. Let $H' = H \setminus S \cap T$ and let $m = |S \cap T|$. Recall that no p.c. cycle spans $[l]$. Thus, there must be an arc between $r \in R$ and $u \in S \cup T$ by (f) and (g). Hence, the base graph of H' are completely bipartite. Moreover, there is an arc from $u \in S \cap T$ to $r \in R$, so each $r \in R$ has in degree at most $\Delta_-(H) - m$ in H' . Also, each $u \in S \cup T \setminus (S \cap T)$ has in degree at most $\Delta_-(H)$ in H' . Therefore, by (e)

$$\begin{aligned}
 |R|(|S| + |T| - 2m) &\leq (\Delta_-(H) - m)|R| + \Delta_-(H)(|S| + |T| - 2m) \\
 |R|(|R| - 2m) &\leq (\Delta_-(H) - m)|R| + \Delta_-(H)(|R| - 2m) \\
 \Delta_-(H) &\geq |R|/2 \geq (2(k-1)\delta - l)/8,
 \end{aligned}$$

so there exists a vertex $x \in R \cup S \cup T$ with in-degree at least $(2(k-1)\delta - l)/8$. Suppose $x \in R$. This means that

$$|\{u \in S \cup T : (x, u) \notin E(G) \text{ or } c(x, u) = c(x, x+1)\}| \geq (2d - l)/8. \quad (28)$$

Since P is a p.c. path of maximal length, each vertex $j \in V(G)$ such that (x, j) is an edge and $c(x, j) \neq c(x, x+1)$ must be in $[l]$ by (f). There are at least $(k-1)\delta$ such vertices as $\delta_k^{mon}(G) = \delta$. Therefore, together with (28) we have

$$l \geq |\{x\}| + (k-1)\delta + (2(k-1)\delta - l)/8 = 1 + (10(k-1)\delta - l)/8,$$

so $l \geq (10(k-1)\delta + 8)/9$, a contradiction. A similar argument holds if $x \in S \cup T$. \square

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